

Nowhere differentiable functions with respect to the position

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Abstract

Let Ω be a bounded domain in \mathbb{C} such that $\partial\Omega$ does not contain isolated points. Let $R(\Omega)$ be the space of uniform limits on $\overline{\Omega}$ of rational functions with poles off $\overline{\Omega}$, endowed with the supremum norm. We prove that either generically all functions f in $R(\Omega)$ satisfy

$$\limsup_{\substack{z \rightarrow z_0 \\ z \in \partial\Omega}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty$$

for every $z_0 \in \partial\Omega$ or no such function in $R(\Omega)$ meets this requirement. In the first case, the generic function $f \in R(\Omega)$ is nowhere differentiable on $\partial\Omega$ with respect to the position. We give specific examples where each case of the previous dichotomy holds. We also extend the previous result to unbounded domains.

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1 Introduction

It is well known that the Weierstrass function $u : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$u(x) = \sum_{n=0}^{+\infty} a^n \cos(b^n x)$$

for $a \in (0, 1)$, b an odd integer and $ab > 1 + \frac{3\pi}{2}$ is a 2π - periodic function which is continuous and nowhere differentiable. Furthermore, this phenomenon is generic in the spaces $C([0, 1])$, $C(\mathbb{R})$ and $C(\mathbb{R}/2\pi)$. In [2] and

[1] the Weierstrass function was complexified, giving a 2π - periodic function of analytic type and the phenomenon was proven to be generic in the disc algebra $A(D)$, where $D = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disc (centered at 0). We remind that a function $f : \overline{D} \rightarrow \mathbb{C}$ belongs to $A(D)$ if and only if f is continuous on \overline{D} and holomorphic in D . More precisely, there exist functions $f \in A(D)$ such that the (real) functions $Re f(e^{i\theta})$ and $Im f(e^{i\theta})$ are nowhere differentiable with respect to the real parameter θ . Furthermore, the set of these functions $f \in A(D)$ is residual in $A(D)$, where the space $A(D)$ is endowed with the supremum norm on \overline{D} ; it contains the set of functions $f \in A(D)$ such that

$$\limsup_{t \rightarrow t_0} \left| \frac{Re f(e^{it}) - Re f(e^{it_0})}{t - t_0} \right| = \limsup_{t \rightarrow t_0} \left| \frac{Im f(e^{it}) - Im f(e^{it_0})}{t - t_0} \right| = +\infty \quad (*)$$

for all $t_0 \in \mathbb{R}$. The last set is G_δ - dense in $A(D)$.

In [5] a generalization of the previous result is obtained, where the open unit disc D is replaced by a domain Ω , where Ω is bounded by a finite set of disjoint Jordan curves. Now, the parametrization of each bounded Jordan curve is induced by a conformal Riemann mapping. It is essential that the denominator in relation (*) is the quantity $t - t_0$, where t is the (real) parameter of the curve $\gamma : t \mapsto \gamma(t)$; thus, the functions obtained in this way are nowhere differentiable with respect to the real parameter t .

In the present paper we wish to obtain analogous results, where the functions will be nowhere differentiable with respect to the position $z \in \mathbb{C}$. It suffices to prove that it holds

$$\limsup_{\substack{z \rightarrow z_0 \\ z \in \partial\Omega}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty \quad (**)$$

for all (or some) $z_0 \in \partial\Omega$. Certainly, if the boundary of Ω is a curve with a smooth parametrization with non - vanishing derivative, then relation (*) implies relation (**). This yields that the phenomenon of functions $f \in A(\Omega)$ satisfying relation (**) for every $z_0 \in J$, where $J \subseteq \partial\Omega$ is a compact set without isolated points, is residual in $A(\Omega)$; here Ω is a disc, a half plane or the complement of a disc or even other domains related to the previous ones. Taking advantage of Baire's Category Theorem we prove, for instance, that this phenomenon is generic when Ω is a bounded angular sector. Several such examples are given in Section 4 below.

The main result is proven in Section 2 (Theorem 2.1). It states that if Ω is a bounded domain in \mathbb{C} , $J \subseteq \partial\Omega$ is a compact set without isolated points

and $S(\Omega, J)$ denotes the set of functions $f \in R(\Omega)$, satisfying

$$\limsup_{\substack{z \rightarrow z_0 \\ z \in J}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty \quad (***)$$

for all $z_0 \in J$, then the class $S(\Omega, J)$ is either void or G_δ -dense in $R(\Omega)$. Here, $R(\Omega)$ denotes the space of uniform limits on $\overline{\Omega}$ of rational functions with poles off $\overline{\Omega}$, endowed with the supremum norm on $\overline{\Omega}$. Quite often, it holds $R(\Omega) = A(\Omega)$; see the relevant comments in Section 3.

In order to generalize the previous result to unbounded domains Ω , V. Nestoridis suggested to replace $R(\Omega)$ with the space $\tilde{R}(\Omega)$. The space $\tilde{R}(\Omega)$ consists of all functions $f : \overline{\Omega} \rightarrow \mathbb{C}$, where the closure $\overline{\Omega}$ is taken in \mathbb{C} and apparently does not contain ∞ , which are uniform limits on each compact subset of $\overline{\Omega}$ of rational functions with poles off $\overline{\Omega}$. This is a Fréchet space, endowed with the seminorms

$$\sup_{\substack{z \in \overline{\Omega} \\ |z| \leq m}} |f(z)|, f \in \tilde{R}(\Omega) \text{ for } m = 1, 2, \dots$$

Then again the class $S(\Omega, J)$ is either void or G_δ -dense in $\tilde{R}(\Omega)$. This is the content of Section 3. Also, quite often it holds $\tilde{R}(\Omega) = A(\Omega)$.

In Section 4 several examples are given. In one such example it holds $S(\Omega, J) = \emptyset$, but we do not have an analogous example if in addition it holds $\overline{\Omega}^\circ = \Omega$. In every other example in Section 4 it holds $S(\Omega, J) \neq \emptyset$ and the generic function f in $A(\Omega)$ is nowhere differentiable with respect to the position on $\partial\Omega$.

2 A dichotomy result for bounded domains

Let Ω be a bounded domain in \mathbb{C} . We denote by $R(\Omega)$ the set of uniform limits on $\overline{\Omega}$ of rational functions with poles off $\overline{\Omega}$. The space $R(\Omega)$ is a Banach space endowed with the supremum norm on $\overline{\Omega}$.

More generally, if $K \subseteq \mathbb{C}$ is a compact set, then $R(K)$ is the set of uniform limits on K of rational functions with poles off K , endowed with the supremum norm on K . The space $R(K)$ is also a Banach space. Obviously, it holds $R(\Omega) = R(\overline{\Omega})$ for every bounded domain $\Omega \subseteq \mathbb{C}$.

Let $\Omega \subseteq \mathbb{C}$ be a bounded domain and $J \subseteq \partial\Omega$ be a compact set without isolated points. We denote with $S(\Omega, J)$ the following class of functions

$$S(\Omega, J) = \{f \in R(\Omega) : \limsup_{\substack{z \rightarrow z_0 \\ z \in J \setminus \{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty \text{ for every } z_0 \in J\}.$$

Theorem 2.1. Under the above assumptions and notations, the class $S(\Omega, J)$ is either void or G_δ - dense in $R(\Omega)$.

Proof. We suppose that it holds $S(\Omega, J) \neq \emptyset$ and let $f \in S(\Omega, J)$. We denote with E_n the following set

$$E_n = \{g \in R(\Omega) : \text{for every } z_0 \in J \text{ there exists a } z \in (J \setminus \{z_0\}) \cap D(z_0, \frac{1}{n})$$

$$\text{such that } \left| \frac{f(z) - f(z_0)}{z - z_0} \right| > n\}$$

where $D(z_0, \frac{1}{n})$ denotes the open disc centered at z_0 and with radius $\frac{1}{n}$. Obviously, it holds

$$S(\Omega, J) = \bigcap_{n=1}^{+\infty} E_n.$$

Firstly we prove that each E_n is an open subset of $(R(\Omega), \|\cdot\|_\infty)$, or equivalently, that each $R(\Omega) \setminus E_n$ is a closed set. Indeed, let $\{g_m\}_{m \geq 1} \subseteq R(\Omega) \setminus E_n$ and $g \in R(\Omega)$ such that $g_m \rightarrow g$. Then, for every $m \geq 1$, there exists a $z_m \in J$ satisfying

$$\left| \frac{g_m(z) - g_m(z_m)}{z - z_m} \right| \leq n$$

for every $z \in (J \setminus \{z_m\}) \cap D(z_m, \frac{1}{n})$. Since J is a compact set, there exists a subsequence of $\{z_m\}_{m \geq 1}$ which converges to a single point $z_0 \in J$. Without loss of generality, we may assume that $\{z_m\}_{m \geq 1}$ converges to z_0 . Let $z \in (J \setminus \{z_0\}) \cap D(z_0, \frac{1}{n})$ be a fixed point. Then, there exists an index $m_0 \in \mathbb{N}$ satisfying $z \in (J \setminus \{z_m\}) \cap D(z_m, \frac{1}{n})$ for every $m \geq m_0$. Consequently, for every $m \geq m_0$ it holds

$$\begin{aligned} |g(z) - g(z_0)| &\leq |g(z) - g_m(z)| + |g_m(z) - g_m(z_m)| + |g_m(z_m) - g(z_0)| \leq \\ &\leq |g(z) - g_m(z)| + |g_m(z_m) - g(z_0)| + n|z_m - z|. \end{aligned}$$

Therefore, by taking limits in the previous relation as $m \rightarrow +\infty$ we obtain that it holds $|g(z) - g(z_0)| \leq n|z - z_0|$ for every $z \in (J \setminus \{z_0\}) \cap D(z_0, \frac{1}{n})$, because $g_m \rightarrow g$ uniformly and $z_m \rightarrow z_0$.

Thus, $g \in R(\Omega) \setminus E_n$ and as a result, E_n is a closed set. Therefore, its complement is an open set for every $n \geq 1$.

It remains to prove the density of $S(\Omega, J)$. Let $g \in R(\Omega)$. Then, there exists a rational function $q \equiv q_\varepsilon$ with poles off $\overline{\Omega}$ such that $\|(g - f) - q\|_\infty < \varepsilon$ for a fixed $\varepsilon > 0$. Since q' is continuous on $\overline{\Omega}$, there exists $M < +\infty$ satisfying

$\|q'\|_\infty \leq M$. Considering a fixed point $z_0 \in J$ we can find a sequence $\{z_m\}_{m \geq 1}$ in $J \setminus \{z_0\}$ such that $z_m \rightarrow z_0$ satisfying

$$\lim_{m \rightarrow +\infty} \left| \frac{f(z_m) - f(z_0)}{z_m - z_0} \right| = +\infty.$$

By the triangle inequality, it holds

$$\left| \frac{(f+q)(z_m) - (f+q)(z_0)}{z_m - z_0} \right| \geq \left| \frac{f(z_m) - f(z_0)}{z_m - z_0} \right| - \left| \frac{q(z_m) - q(z_0)}{z_m - z_0} \right|.$$

At the same time, it also holds

$$\lim_{m \rightarrow +\infty} \left| \frac{q(z_m) - q(z_0)}{z_m - z_0} \right| = |q'(z_0)| \leq M.$$

Therefore, by combining the previous two relations, we obtain

$$\lim_{m \rightarrow +\infty} \left| \frac{(f+q)(z_m) - (f+q)(z_0)}{z_m - z_0} \right| = +\infty$$

and thus

$$\limsup_{\substack{z \rightarrow z_0 \\ z \in J \setminus \{z_0\}}} \left| \frac{(f+q)(z) - (f+q)(z_0)}{z - z_0} \right| = +\infty$$

for every $z_0 \in J$. Consequently, we deduce that $(f+q) \in \overline{S(\Omega, J)}$. Since $\|g - (f+q)\|_\infty < \varepsilon$ and $\varepsilon > 0$ is arbitrary, it follows that $g \in \overline{S(\Omega, J)}$. Thus, we have proved the density of $S(\Omega, J)$. Baire's Theorem completes the proof. \blacksquare

Let $K \subseteq \mathbb{C}$ be a compact set. Then, we denote with $A(K)$ the set of all functions $f : K \rightarrow \mathbb{C}$ which are continuous on K and holomorphic in K° (if $K^\circ = \emptyset$, then $A(K) = C(K)$). We endow the space $A(K)$ with the supremum norm on K and thus, $A(K)$ becomes a Banach space. If Ω is a bounded domain in \mathbb{C} , then $R(\Omega)$ is a closed subspace of $A(\overline{\Omega})$.

Definition 2.2 (Sets of approximation). We say that a compact set $K \subseteq \mathbb{C}$ is a set of approximation if it holds $A(K) = R(K)$.

A characterization of whether a compact set K is a set of approximation is given in [7], using the notion of continuous analytic capacity. Furthermore, sufficient conditions ensuring that a compact set K is a set of approximation are the following:

- (i) $(\mathbb{C} \cup \{\infty\}) \setminus K$ has finitely many connected components ([6], exercise 1, chapter 20, page 394)

- (ii) The diameters of the connected components of the complement of K (even though there may be infinitely many such components) are uniformly bounded away from zero ([3]).
- (iii) K has area measure zero ([3]).

Remark 2.3. If $\Omega \subseteq \mathbb{C}$ is a bounded domain in \mathbb{C} such that $\overline{\Omega}$ is a compact set of approximation, then Theorem 2.1 ensures that the class $S(\Omega, J)$ is either void or G_δ - dense in $A(\overline{\Omega}) = R(\Omega)$. We define $A(\Omega)$ to be the class of functions $f : \overline{\Omega} \rightarrow \mathbb{C}$ continuous on $\overline{\Omega}$ and holomorphic in Ω , endowed with the supremum norm on $\overline{\Omega}$. This is also a Banach space and $A(\overline{\Omega}) \subseteq A(\Omega)$ but we do not always have the equality $A(\overline{\Omega}) = A(\Omega)$. If $(\overline{\Omega})^\circ = \Omega$; that is, if Ω is a Carathéodory domain, then we have the equality $A(\overline{\Omega}) = A(\Omega)$ and in this case Theorem 2.1 ensures that $S(\Omega, J)$ is either void or G_δ - dense in $A(\Omega)$.

3 Unbounded domains

Let $E \subseteq \mathbb{C}$ be an unbounded open set. We denote with $\tilde{R}(E)$ the set of all functions which are uniform limits on each compact subset of \overline{E} of rational functions with poles off \overline{E} . The natural topology of $\tilde{R}(E)$ is the topology of uniform convergence on each compact subset of \overline{E} . Equivalently, it is defined by the sequence of seminorms

$$\sup_{\substack{z \in \overline{E} \\ |z| \leq n}} |f(z)|, f \in \tilde{R}(E) \text{ for } n = 1, 2, \dots$$

Moreover the space $\tilde{R}(E)$ endowed with these seminorms is a Fréchet space.

Theorem 3.1. Let $\Omega \subseteq \mathbb{C}$ be an unbounded domain and $J \subseteq \partial\Omega$ be a compact set without isolated points. Then, the class

$$S(\Omega, J) = \{f \in \tilde{R}(\Omega) : \limsup_{\substack{z \rightarrow z_0 \\ z \in J \setminus \{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty \text{ for every } z_0 \in J\}$$

is either void or G_δ - dense in $\tilde{R}(\Omega)$.

Proof. We suppose that $S(\Omega, J) \neq \emptyset$ and let $f \in S(\Omega, J)$. We denote with E_n the following set

$$E_n = \{g \in \tilde{R}(\Omega) : \text{for every } z_0 \in J \text{ there exists a } z \in (J \setminus \{z_0\}) \cap D(z_0, \frac{1}{n})\}$$

$$\text{such that } \left| \frac{g(z) - g(z_0)}{z - z_0} \right| > n\}$$

Obviously, it holds

$$S(\Omega, J) = \bigcap_{n=1}^{+\infty} E_n.$$

Firstly we show that each E_n is an open subset of $\tilde{R}(\Omega)$, or equivalently, that each $\tilde{R}(\Omega) \setminus E_n$ is closed in $\tilde{R}(\Omega)$. Indeed, let $\{g_m\}_{m \geq 1}$ be a sequence of functions in $\tilde{R}(\Omega) \setminus E_n$ which converges uniformly on the compact subsets of $\overline{\Omega}$ to a function $g \in \tilde{R}(\Omega)$. Then, for every $m \geq 1$, there exists a $z_m \in J$ satisfying

$$\left| \frac{g_m(z) - g_m(z_m)}{z - z_m} \right| \leq n$$

for every $z \in (J \setminus \{z_m\}) \cap D(z_m, \frac{1}{n})$. Since J is a compact set, there exists a subsequence of $\{z_m\}_{m \geq 1}$ which converges to a single point $z_0 \in J$. Without loss of generality, we may assume that $\{z_m\}_{m \geq 1}$ converges to z_0 . Let $z \in (J \setminus \{z_0\}) \cap D(z_0, \frac{1}{n})$ be a fixed point. Then, there exists an index $m_0 \in \mathbb{N}$ satisfying $z \in (J \setminus \{z_m\}) \cap D(z_m, \frac{1}{n})$ for every $m \geq m_0$. Consequently, for every $m \geq m_0$ it holds

$$\begin{aligned} |g(z) - g(z_0)| &\leq |g(z) - g_m(z)| + |g_m(z) - g_m(z_m)| + |g_m(z_m) - g(z_0)| \leq \\ &\leq |g(z) - g_m(z)| + |g_m(z_m) - g(z_0)| + n|z_m - z|. \end{aligned}$$

Furthermore, the sequence $\{g_m\}_{m \geq 1}$ converges uniformly to $g \in \tilde{R}(\Omega)$ on J , since $J \subseteq \overline{\Omega}$ is a compact set. Therefore, by taking limits in the previous relation as $m \rightarrow +\infty$ we obtain that $|g(z) - g(z_0)| \leq n|z - z_0|$ for every $z \in (J \setminus \{z_0\}) \cap D(z_0, \frac{1}{n})$.

Thus $g \in \tilde{R}(\Omega) \setminus E_n$ and as a result, E_n is a closed set. Therefore, its complement is an open set for every $n \geq 1$.

It remains to prove the density of $S(\Omega, J)$. Let $g \in \tilde{R}(\Omega)$. Then, there exists a sequence of rational functions $\{q_m\}_{m \geq 1}$ with poles off $\overline{\Omega}$ which converges uniformly to the function $g - f$ on each compact subset of $\overline{\Omega}$. Obviously, the sequence $\{f + q_m\}_{m \geq 1}$ converges uniformly to g on the compact subsets of $\overline{\Omega}$. Now, let q be a rational function with poles off $\overline{\Omega}$. Since q' is continuous on the compact set $J \subseteq \partial\Omega$, there exists a $M < +\infty$ such that $|q'(z)| \leq M$ for every $z \in J$. Let $z_0 \in J$; since it holds

$$\limsup_{\substack{z \rightarrow z_0 \\ z \in J \setminus \{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty$$

we can find a sequence $\{z_m\}_{m \geq 1}$ such that $z_m \in (J \setminus \{z_0\}) \cap B(z_0, \frac{1}{n})$ for every $m \geq 1$, also $z_m \rightarrow z_0$ and

$$\lim_{m \rightarrow +\infty} \left| \frac{f(z_m) - f(z_0)}{z_m - z_0} \right| = +\infty.$$

At the same time, it also holds

$$\left| \frac{(f+q)(z_m) - (f+q)(z_0)}{z_m - z_0} \right| \geq \left| \frac{f(z_m) - f(z_0)}{z_m - z_0} \right| - \left| \frac{q(z_m) - q(z_0)}{z_m - z_0} \right|$$

and obviously it holds

$$\lim_{m \rightarrow +\infty} \left| \frac{q(z_m) - q(z_0)}{z_m - z_0} \right| = |q'(z_0)| \leq M.$$

Therefore, by combining the previous two relations, we obtain

$$\lim_{m \rightarrow +\infty} \left| \frac{(f+q)(z_m) - (f+q)(z_0)}{z_m - z_0} \right| = +\infty$$

and thus

$$\limsup_{\substack{z \rightarrow z_0 \\ z \in J \setminus \{z_0\}}} \left| \frac{(f+q)(z) - (f+q)(z_0)}{z - z_0} \right| = +\infty$$

for every $z_0 \in J$. Therefore, we can deduce that $(f+q_n) \in S(\Omega, J)$ for every $n \geq 1$ and thus, since $f+q_n \rightarrow g$ in the topology of $\tilde{R}(E)$, it follows that $g \in \underline{S(\Omega, J)}$. Thus, we have proved the density of $S(\Omega, J)$. Baire's Theorem completes the proof. ■

If $E \subseteq \mathbb{C}$ is a closed set, then by $A(E)$ we denote the set of all functions $f : E \rightarrow \mathbb{C}$ which are continuous on E and holomorphic in E° (if $E^\circ = \emptyset$, then $A(E) = C(E)$). In any case, $A(E)$ is a Fréchet space when endowed with the seminorms

$$\sup_{\substack{z \in E \\ |z| \leq n}} |f(z)|, f \in A(E) \text{ for } n = 1, 2, \dots$$

In addition, if $\Omega \subseteq \mathbb{C}$ is an unbounded domain and $\overline{\Omega} \cap \overline{D(0, n)}$ is a compact set of approximation for every $n \geq 1$, then $\tilde{R}(\Omega) = A(\overline{\Omega})$. Indeed, obviously it holds that $\tilde{R}(\Omega) \subseteq A(\overline{\Omega})$. Let $g \in A(\overline{\Omega})$ and $n \geq 1$. Since $\overline{\Omega} \cap \overline{D(0, n)}$ is a compact set of approximation, there exists a rational function ϕ_n with poles off $X_n = \overline{\Omega} \cap \overline{D(0, n)}$, such that

$$\sup_{z \in X_n} |g(z) - \phi_n(z)| < \frac{1}{2n}.$$

The function ϕ_n is holomorphic in an open set containing the compact set X_n . Now, let $B \subseteq (\mathbb{C} \cup \{\infty\}) \setminus X_n$ be a set that intersects each component of $(\mathbb{C} \cup \{\infty\}) \setminus X_n$. We also assume that the set B satisfies $B \cap L = \{\infty\}$, where L is the component of $(\mathbb{C} \cup \{\infty\}) \setminus X_n$ containing ∞ . From Runge's theorem, which can be applied because ϕ_n is holomorphic in an open set containing X_n , we can find a rational function g_n with poles in B such that

$$\sup_{X_n} |\phi_n(z) - g_n(z)| < \frac{1}{2n}.$$

Obviously, $\|g - g_n\|_\infty < \frac{1}{n}$ on $X_n = \overline{\Omega} \cap \overline{B(0, n)}$. Now, let r be a pole of g_n . There are two possible cases.

- If $r \in \overline{D(0, n)}$, then $r \notin \overline{\Omega}$ since $r \notin X_n = \overline{\Omega} \cap \overline{D(0, n)}$.
- If $r \notin \overline{D(0, n)}$, then $r \in [(\mathbb{C} \cup \{\infty\}) \setminus X_n] \cap L$ and thus, $r = \infty \notin \overline{\Omega}$.

In every case, we have that it holds $r \notin \overline{\Omega}$. Consequently, for every $n \geq 1$, there exists a rational function g_n with poles off $\overline{\Omega}$ such that $|g(z) - g_n(z)| < \frac{1}{n}$ for every $z \in \overline{\Omega} \cap \overline{D(0, n)}$. Moreover, the sequence $\{g_n\}_{n \geq 1}$ converges uniformly to g on the compact subsets of $\overline{\Omega}$ and thus, $g \in \tilde{R}(\Omega)$. Therefore, we obtain that it holds $\tilde{R}(\Omega) = A(\overline{\Omega})$.

However, we do not know if the converse holds; that is, if for an unbounded domain $\Omega \subseteq \mathbb{C}$ it holds $\tilde{R}(\Omega) = A(\overline{\Omega})$, is it necessarily true that $\overline{\Omega} \cap \overline{D(0, \lambda_n)}$ is a compact set of approximation for all elements λ_n of a sequence $\{\lambda_n\}_{n \geq 1} \subseteq \mathbb{R}$ converging to $+\infty$?

4 Examples

In this section we give specific examples. In each such example we clarify if the class $S(\Omega, J)$ is void or G_δ -dense.

Example 4.1. Let $\Omega = D$ be the open unit disc and $\mathbb{T} = \partial D$. We consider the function $g : D \rightarrow \mathbb{C}$ defined as follows

$$g(z) = \sum_{n=0}^{+\infty} a^n z^{b^n} \text{ for every } z \in D.$$

where $a \in (0, 1)$, b an odd integer and $ab > 1 + \frac{3\pi}{2}$. Then, it is easy to see that it holds $Re(g) = u$, where u is the Weierstrass function (see also [1], [2],

[5]). It is known that it holds $g \in A(D)$. In addition,

$$\limsup_{\substack{\theta \rightarrow \theta_0 \\ \theta \in (\theta_0 - 1, \theta_0 + 1) \setminus \{\theta_0\}}} \left| \frac{g(e^{i\theta}) - g(e^{i\theta_0})}{\theta - \theta_0} \right| = +\infty$$

for every $\theta_0 \in [0, 2\pi]$. We consider the following class of functions

$$S(D, \mathbb{T}) = \{f \in A(D) : \limsup_{\substack{z \rightarrow z_0 \\ z \in \mathbb{T} \setminus \{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty \text{ for every } z_0 \in \mathbb{T}\}.$$

Then, since $A(D) = R(\overline{D})$ it either holds $S(D, \mathbb{T}) = \emptyset$ or $S(D, \mathbb{T})$ is G_δ -dense in $(A(\Omega), \|\cdot\|_\infty)$, according to Theorem 2.1. Let $z_0 = e^{i\theta_0} \in \mathbb{T}$. We consider a sequence $\{\theta_n\}_{n \geq 1}$ in $\mathbb{R} \setminus \{\theta_0\}$ with $\theta_n \rightarrow \theta_0$ satisfying

$$\lim_{n \rightarrow +\infty} \left| \frac{g(e^{i\theta_n}) - g(e^{i\theta_0})}{\theta_n - \theta_0} \right| = +\infty.$$

In addition, for every $n \geq 1$ we set $z_n = e^{i\theta_n} \in \mathbb{T}$. Then, $z_n \rightarrow z_0$ and there exists an index $n_0 \in \mathbb{N}$ such that $z_n \neq z_0$ for every $n \geq n_0$. For every $n \geq n_0$ we have

$$\frac{g(z_n) - g(z_0)}{z_n - z_0} = \frac{g(e^{i\theta_n}) - g(e^{i\theta_0})}{\theta_n - \theta_0} \cdot \frac{\theta_n - \theta_0}{e^{i\theta_n} - e^{i\theta_0}}.$$

Since

$$\lim_{n \rightarrow +\infty} \frac{\theta_n - \theta_0}{e^{i\theta_n} - e^{i\theta_0}} = \frac{1}{ie^{i\theta_0}} \neq 0$$

we obtain that it holds

$$\lim_{n \rightarrow +\infty} \left| \frac{g(z_n) - g(z_0)}{z_n - z_0} \right| = +\infty$$

and therefore

$$\limsup_{\substack{z \rightarrow z_0 \\ z \in \mathbb{T} \setminus \{z_0\}}} \left| \frac{g(z) - g(z_0)}{z - z_0} \right| = +\infty$$

for every $z_0 \in \mathbb{T}$. The previous relation implies that $g \in S(D, \mathbb{T})$ and thus $S(D, \mathbb{T}) \neq \emptyset$; according to Theorem 2.1, the class $S(D, \mathbb{T})$ is G_δ -dense in $A(D)$.

Example 4.2. Let $\Omega = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$. For every $n \geq 1$ we consider the following classes of functions

$$S(\Omega, J_n) = \{f \in A(\Omega) : \limsup_{\substack{z \rightarrow z_0 \\ z \in J_n \setminus \{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty \text{ for every } z_0 \in J_n\}$$

where $J_n = [-in, +in]$. In addition, let

$$S(\Omega, J) = \{f \in A(\Omega) : \limsup_{\substack{z \rightarrow z_0 \\ z \in J \setminus \{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty \text{ for every } z_0 \in J\}$$

where

$$J = \bigcup_{n=1}^{+\infty} J_n = i\mathbb{R}.$$

Then, the class $S(\Omega, J)$ is G_δ - dense in $A(\Omega)$, where the space $A(\Omega)$ is endowed with the topology of uniform convergence on the compact subsets of $\overline{\Omega}$.

Proof. We prove that each $S(\Omega, J_n)$ is G_δ - dense in $A(\Omega) = \widetilde{R}(\Omega)$. Since $(\mathbb{C} \cup \{\infty\}) \setminus (\overline{\Omega} \cap D(0, n))$ is a connected set, it follows that $A((\overline{\Omega})^\circ \cap D(0, n)) = R(\overline{\Omega} \cap D(0, n))$ for every $n \geq 1$ and thus, according to Section 3, it is enough to prove that each $S(\Omega, J_n)$ is non - void.

We consider the entire function $\phi : \mathbb{C} \rightarrow \mathbb{C}$ satisfying $\phi(w) = e^{-w}$ for every $w \in \mathbb{C}$. Obviously, $\phi'(w) = -e^{-w} \neq 0$ for every $w \in \mathbb{C}$. We also consider the function $f : \Omega \rightarrow \mathbb{C}$ with $f = g \circ \phi$, where g is the function defined in Example 4.1. Obviously, $\phi(\Omega) \subseteq D(0, 1)$ and $\phi(J_n) \subseteq \mathbb{T}$. The reader can easily verify that it holds $f \in S(\Omega, J_n)$ for every $n \geq 1$. Thus, the class $S(\Omega, J_n)$ is G_δ - dense in $A(\Omega)$ and since it holds

$$S = \bigcap_{n=1}^{+\infty} S(\Omega, J_n)$$

Baire's Theorem implies that the class $S(\Omega, J)$ is G_δ - dense in $A(\Omega)$. ■

Remark 4.3. Example 4.2 shows that in the definition of the class $S(\Omega, J)$ in Section 3 the set J may not be a compact one.

Example 4.4. We consider the following sets

$$A = \{re^{i\frac{3\pi}{4}} : 0 \leq r \leq 1\}$$

$$B = \{e^{i\theta} : \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}\}$$

and

$$C = \{re^{i\frac{\pi}{4}} : 0 \leq r \leq 1\}.$$

Let Ω be the Jordan domain bounded by $A \cup B \cup C$. It clearly holds $\partial\Omega = A \cup B \cup C$. Then, the class of functions

$$S(\Omega, \partial\Omega) = \{f \in A(\Omega) : \limsup_{\substack{z \rightarrow z_0 \\ z \in \partial\Omega \setminus \{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty \text{ for every } z_0 \in \partial\Omega\}.$$

is G_δ - dense in $(A(\Omega), \|\cdot\|_\infty)$.

Proof. We consider the following classes of functions

$$S(\Omega, A) = \{f \in A(\Omega) : \limsup_{\substack{z \rightarrow z_0 \\ z \in A \setminus \{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty \text{ for every } z_0 \in A\}$$

$$S(\Omega, B) = \{f \in A(\Omega) : \limsup_{\substack{z \rightarrow z_0 \\ z \in B \setminus \{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty \text{ for every } z_0 \in B\}$$

$$S(\Omega, C) = \{f \in A(\Omega) : \limsup_{\substack{z \rightarrow z_0 \\ z \in C \setminus \{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty \text{ for every } z_0 \in C\}.$$

Obviously, $S(\Omega, A) \cap S(\Omega, B) \cap S(\Omega, C) \subseteq S(\Omega, \partial\Omega)$. From Example 4.2, there exists a function $h \in A(R)$ (where R is the open right half plane) satisfying

$$\limsup_{\substack{z \rightarrow z_0 \\ z \in i\mathbb{R} \setminus \{z_0\}}} \left| \frac{h(z) - h(z_0)}{z - z_0} \right| = +\infty$$

for every $z_0 \in i\mathbb{R}$. In the same way, one can prove that there exists a function $\phi \in A(L)$ (where L is the open left half plane) satisfying

$$\limsup_{\substack{z \rightarrow z_0 \\ z \in i\mathbb{R} \setminus \{z_0\}}} \left| \frac{\phi(z) - \phi(z_0)}{z - z_0} \right| = +\infty$$

for every $z_0 \in i\mathbb{R}$.

We consider the functions $\omega_1, \omega_2 \in A(\Omega)$ with

$$\omega_1(z) = h(e^{-i\frac{\pi}{4}}z)$$

and

$$\omega_2(z) = \phi(e^{i\frac{\pi}{4}}z)$$

for every $z \in \Omega$. We have proved that $\omega_1 \in S(\Omega, A)$ and $\omega_2 \in S(\Omega, C)$; thus, according to Section 2, the classes $S(\Omega, A)$ and $S(\Omega, C)$ are G_δ - dense in $A(\Omega)$. In addition, if g is the function defined in Example 4.1, then $(g|_\Omega) \in$

$S(\Omega, B)$ and therefore, the class $S(\Omega, B)$ is also G_δ - dense in $A(\Omega)$. According to Baire's Theorem, it follows that the class $S(\Omega, A) \cap S(\Omega, B) \cap S(\Omega, C)$ is G_δ - dense in $A(\Omega)$. Since $S(\Omega, A) \cap S(\Omega, B) \cap S(\Omega, C) \subseteq S(\Omega, J)$, we obtain that it holds $S(\Omega, J) \neq \emptyset$ and thus, the class $S(\Omega, J)$ is also G_δ - dense in $A(\Omega)$. ■

Example 4.5. Let Ω be the interior of a convex polygonal domain, with

$$\partial\Omega = [a_0, a_1] \cup [a_1, a_2] \cup \cdots \cup [a_n, a_0].$$

Then, the class of functions

$$S(\Omega, \partial\Omega) = \left\{ f \in A(\Omega) : \limsup_{\substack{z \rightarrow z_0 \\ z \in \partial\Omega \setminus \{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty \right. \\ \left. \text{for every } z_0 \in \partial\Omega \right\}$$

is G_δ - dense in $(A(\Omega), \|\cdot\|_\infty)$.

Proof. We consider the following classes of functions

$$S_i(\Omega, [a_i, a_{i+1}]) = \left\{ f \in A(\Omega) : \limsup_{\substack{z \rightarrow z_0 \\ z \in [a_i, a_{i+1}] \setminus \{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty \right. \\ \left. \text{for every } z_0 \in [a_i, a_{i+1}] \right\}$$

for every $i = 0, 1, \dots, n-1$ and

$$S_n(\Omega, [a_n, a_0]) = \left\{ f \in A(\Omega) : \limsup_{\substack{z \rightarrow z_0 \\ z \in [a_n, a_0] \setminus \{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty \right. \\ \left. \text{for every } z_0 \in [a_n, a_0] \right\}.$$

Let also H_i be the (open) half planes with boundary the lines containing the line segments $[a_i, a_{i+1}]$ for every $i = 0, 1, \dots, n-1$, such that $H_i^\circ \cap \Omega \neq \emptyset$. In the same way, we can also define the set H_n . Thus

$$\Omega = \bigcap_{i=0}^n H_i^\circ.$$

It is easy to prove (in the same way as in Example 4.4) that there exist functions $f_i \in S_i(\Omega, [a_i, a_{i+1}])$ for every $i = 0, 1, \dots, n-1$. Thus, each

$S_i(\Omega, [a_i, a_{i+1}])$ is G_δ - dense in $A(\Omega)$ (if $i = n$, obviously we consider the class $S_n(\Omega, [a_n, a_0])$) and thus, the class

$$\left(\bigcap_{i=0}^{n-1} S_i(\Omega, [a_i, a_{i+1}]) \right) \cap S_n(\Omega, [a_n, a_0]) \subseteq S(\Omega, \partial\Omega)$$

is G_δ - dense in $A(\Omega)$. Therefore, $S(\Omega, \partial\Omega) \neq \emptyset$. It follows that the class $S(\Omega, \partial\Omega)$ is G_δ - dense in $A(\Omega)$. ■

Example 4.6. We consider a family of sets $X_i, i = 1, \dots, N$ where each X_i is either an open disc, or the interior of a half plane. We set

$$\Omega = \bigcap_{i=1}^N X_i$$

and we suppose that it holds $\Omega \neq \emptyset$. Let also $\emptyset \neq J \subseteq \partial\Omega$ be a compact set without isolated points. Then, the class of functions

$$S = S(\Omega, J) = \{f \in A(\Omega) : \limsup_{\substack{z \rightarrow z_0 \\ z \in J \setminus \{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty \text{ for every } z_0 \in J\}$$

is G_δ - dense in $A(\Omega) = A(\overline{\Omega})$.

Proof. We notice that it holds

$$J = \bigcup_{k=1}^m J_k$$

where each $J_k \subseteq \partial\Omega$ is either a line segment or an arc. We consider the following classes of functions

$$S_k = S_k(\Omega, J_k) = \{f \in A(\Omega) : \limsup_{\substack{z \rightarrow z_0 \\ z \in J_k \setminus \{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty \text{ for every } z_0 \in J_k\}.$$

Obviously, it holds

$$\bigcap_{k=1}^m S_k \subseteq S.$$

In the same way as in Example 4.4 we obtain that it holds $S_k(\Omega, J_k) \neq \emptyset$ for every $k = 1, \dots, m$. Therefore, according to Sections 2 and 3, we obtain

that each $S_k(\Omega, J_k)$ is G_δ - dense in $A(\Omega)$. Thus, from Baire's Theorem we obtain that the class

$$\bigcap_{k=1}^m S_k$$

is also G_δ - dense in $A(\Omega)$ which in turn implies that $S(\Omega, J) \neq \emptyset$. It follows that $S(\Omega, J)$ is also G_δ - dense in $A(\Omega)$. ■

Example 4.7. Let Ω be the open set that is the intersection of the interior of a square with side = 2 centered at 0, with the complement of the open unit disc (also centered at 0). Then, the class of functions

$$S(\Omega, \partial\Omega) = \{f \in A(\Omega) : \limsup_{\substack{z \rightarrow z_0 \\ z \in \partial\Omega \setminus \{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty \text{ for every } z_0 \in \partial\Omega\}$$

is G_δ - dense in $A(\Omega)$.

Proof. We notice that it holds $\partial\Omega = \mathbb{T} \cup [a_0, a_1] \cup [a_1, a_2] \cup [a_2, a_3] \cup [a_3, a_0]$, where, of course, the sets $[a_0, a_1]$, $[a_1, a_2]$, $[a_2, a_3]$ and $[a_3, a_0]$ are the four sides of the square. We consider the following classes of functions

$$S_i(\Omega, [a_i, a_{i+1}]) = \{f \in A(\Omega) : \limsup_{\substack{z \rightarrow z_0 \\ z \in [a_i, a_{i+1}] \setminus \{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty \\ \text{for every } z_0 \in [a_i, a_{i+1}]\}$$

for every $i = 0, 1, 2$ and

$$S_3(\Omega, [a_3, a_0]) = \{f \in A(\Omega) : \limsup_{\substack{z \rightarrow z_0 \\ z \in [a_3, a_0] \setminus \{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty \\ \text{for every } z_0 \in [a_3, a_0]\}.$$

Then, since the set $(\mathbb{C} \cup \{\infty\}) \setminus \overline{\Omega}$ has a finite number of connected components, according to [6] it holds $A(\Omega) = R(\overline{\Omega})$.

Using the results of Section 2 and in the same way as in Example 4.4 we obtain that the classes $S_0(\Omega, [a_0, a_1]), \dots, S_3([a_3, a_0])$ are G_δ - dense in $A(\Omega)$.

Also, we consider the following class of functions

$$B(\Omega, \mathbb{T}) = \{f \in A(\Omega) : \limsup_{\substack{z \rightarrow z_0 \\ z \in \mathbb{T} \setminus \{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty \text{ for every } z_0 \in \mathbb{T}\}$$

and let g be the function defined in Example 4.1. We consider the following function $h : (\mathbb{C} \setminus \overline{D(0,1)}) \rightarrow \mathbb{C}$ with $h(w) = g(\frac{1}{w})$ for every w . Then it holds $h \in A(\mathbb{C} \setminus \overline{D(0,1)})$ and is also easy to verify that $(h|_{\Omega}) \in B(\Omega, \mathbb{T})$. Thus, the class $B(\Omega, \mathbb{T})$ is G_{δ} - dense in $A(\Omega)$ and since $S_0([a_0, a_1]) \cap S_1([a_1, a_2]) \cap S_2([a_2, a_3]) \cap S_3([a_3, a_0]) \cap B(\Omega, \mathbb{T}) \subseteq S(\Omega, \partial\Omega)$ it follows that $S(\Omega, \partial\Omega) \neq \emptyset$ and thus, the class $S(\Omega, \partial\Omega)$ is G_{δ} - dense in $A(\Omega)$. ■

Remark 4.8. Notice that in the previous example the set Ω is not a convex one.

Remark 4.9. In the previous example, by considering the function $h(w) = g(\frac{1}{w})$ we obtain that $S(\Omega, \mathbb{T}) \neq \emptyset$, where $\Omega = \{z \in \mathbb{C} : |z| > 1\}$. In addition, it is obvious that if $S(\Omega, J) \neq \emptyset$, where $G \subseteq \Omega$ is an open set and $J \subseteq \partial\Omega \cap \partial G$, then it holds $S(G, J) \neq \emptyset$.

Example 4.10. Let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be an open set and $\emptyset \neq J \subseteq \partial\Omega$ be a compact set without isolated points. Let also $V \subseteq \mathbb{C}$ be an open set with $J \subseteq V$. We suppose that there exists a function $\phi : \Omega \cup V \rightarrow \mathbb{C}$, which is 1 - 1, holomorphic and can be extended in a continuous way to a homeomorphism from $\overline{\Omega} \cup \overline{V}$ to \mathbb{C} . We consider the following classes of functions

$$S_1(\Omega, J) = \{f \in A(\Omega) : \limsup_{\substack{z \rightarrow z_0 \\ z \in J \setminus \{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty \text{ for every } z_0 \in J\}$$

and

$$S_2(G, \tilde{J}) = \{f \in A(G) : \limsup_{\substack{z \rightarrow z_0 \\ z \in \tilde{J} \setminus \{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty \text{ for every } z_0 \in \tilde{J}\}$$

where $G = \phi(\Omega)$ and $\tilde{J} = \phi(J)$. Then, if $S_1(\Omega, J) \neq \emptyset$ then it holds $S_2(\Omega, \tilde{J}) \neq \emptyset$.

Proof. It is well known that the set G is an open one, $\phi'(z) \neq 0$ for every $z \in \Omega \cup V$, the function $\phi^{-1} : \phi(\Omega \cup V) \rightarrow \Omega \cup V$ is a holomorphic function with $(\phi^{-1})'(w) \neq 0$ for every $w \in \phi(\Omega \cup V)$ and ϕ^{-1} can be extended continuously to a 1 - 1 function from $\overline{\phi(\Omega \cup V)}$ to \mathbb{C} . Now let $f \in S_1(\Omega, J)$. We consider the function $h = f \circ \phi^{-1} : \phi(\Omega) \rightarrow \mathbb{C}$. Obviously, it holds $h \in A(G)$. In addition, \tilde{J} is a compact set without isolated points since ϕ is

a homeomorphism. Let $\omega_0 = \phi(z_0) \in \tilde{J}$, for a $z_0 \in J$. Then, there exists a sequence $\{z_n\}_{n \geq 1} \in J \setminus \{z_0\}$ with $z_n \rightarrow z_0$ satisfying

$$\lim_{n \rightarrow +\infty} \left| \frac{f(z_n) - f(z_0)}{z_n - z_0} \right| = +\infty.$$

We set $\omega_n = \phi(z_n) \in \tilde{J} \setminus \{\omega_0\}$ for every $n \geq 1$. Then, it holds $\omega_n \rightarrow \omega_0$. In addition, we obtain

$$\left| \frac{h(\omega_n) - h(\omega_0)}{\omega_n - \omega_0} \right| = \left| \frac{f(z_n) - f(z_0)}{\phi(z_n) - \phi(z_0)} \right| = \left| \frac{f(z_n) - f(z_0)}{z_n - z_0} \right| \cdot \left| \frac{z_n - z_0}{\phi(z_n) - \phi(z_0)} \right|.$$

We have

$$\lim_{n \rightarrow +\infty} \left| \frac{z_n - z_0}{\phi(z_n) - \phi(z_0)} \right| = \frac{1}{|\phi'(z_0)|}$$

with $\phi'(z_0) \neq 0$, since $z_0 \in \Omega \cup V$. Thus,

$$\lim_{n \rightarrow +\infty} \left| \frac{h(\omega_n) - h(\omega_0)}{\omega_n - \omega_0} \right| = +\infty$$

which it turn implies that

$$\limsup_{\substack{\omega \rightarrow \omega_0 \\ \omega \in \tilde{J} \setminus \{\omega_0\}}} \left| \frac{h(\omega) - h(\omega_0)}{\omega - \omega_0} \right| = +\infty$$

for every $\omega_0 \in \tilde{J}$. Thus, $h \in S_2(G, \tilde{J})$ and that completes the proof. ■

Remark 4.11. If the class $S_1(\Omega, J)$ is G_δ - dense, then the class $S_2(G, \tilde{J})$ is also G_δ - dense, since the function $\psi : A(\Omega) \rightarrow A(\phi(\Omega))$ with $\psi(f) = f \circ \phi^{-1}$ for every $f \in A(\Omega)$ is a homeomorphism, where the spaces $A(\Omega)$ and $A(\phi(\Omega))$ are endowed with their natural topologies. Notice that $\psi(S_1(\Omega, J)) = S_2(G, \tilde{J})$.

Especially, if D is the open unit disc, $J = \mathbb{T}$, $V = D(0, r)$ with $r > 1$ and we consider the function $\phi : D \cup V \equiv V \rightarrow \mathbb{C}$ which is 1 - 1 and holomorphic, then the respective class $S_2(G, \tilde{J})$ in $A(\phi(D))$ with $\tilde{J} = \phi(\partial D)$ is G_δ - dense in $A(\phi(D))$ with \tilde{J} being not - necessarily a line segment, an arc or the union of the previous two.

Example 4.12. We consider the open set $\Omega = D \setminus [0, \frac{1}{2}]$. Then the class of functions

$$S(\Omega, \partial\Omega) = \{f \in A(\Omega) : \limsup_{\substack{\omega \rightarrow \omega_0 \\ \omega \in \partial\Omega \setminus \{\omega_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty \text{ for every } z_0 \in \partial\Omega\}$$

is void.

Proof. Let $f \in A(\Omega)$. Then the function f is continuous on \overline{D} and holomorphic in $D \setminus \mathbb{R}$. From a known corollary of Morera's Theorem, it follows that f is also holomorphic in D . Thus,

$$\limsup_{\substack{z \rightarrow z_0 \\ z \in [0, \frac{1}{2}] \setminus \{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = |f'(z_0)| < +\infty$$

for every $z_0 \in [0, \frac{1}{2}]$, therefore $f \notin S(\Omega, \partial\Omega)$. It follows that $S(\Omega, \partial\Omega) = \emptyset$. ■

Remark 4.13. From the previous example we deduce that the class $S(\Omega, \partial\Omega)$ is not always G_δ - dense, even if the set Ω is a bounded set.

Example 4.14. We consider the open set $\Omega = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\} \setminus [1, +\infty)$ and the respective class $S(\Omega, J)$, where $J = [1, +\infty)$. Then it holds $S(\Omega, J) = \emptyset$.

Proof. Let $f \in A(\Omega)$. In the same way as in the previous example, the function f can be extended holomorphically on $H = \{z : \operatorname{Re}(z) > 0\}$. Thus

$$\limsup_{\substack{z \rightarrow z_0 \\ z \in J \setminus \{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = |f'(z_0)| < +\infty$$

for every $z_0 \in J$. Therefore, it holds $f \notin S(\Omega, J)$ and thus $S(\Omega, J) = \emptyset$. ■

Remark 4.15. The previous example shows that the class $S(\Omega, J)$ can be also void even if the set Ω is an unbounded one and the set J is not necessarily a compact one.

Remark 4.16. So far, we do not have an example where it holds $S(\Omega, J) = \emptyset$ and $J \subseteq \partial(\overline{\Omega})$.

Example 4.17. So far, we have seen that if $\phi : D(0, r) \rightarrow \mathbb{C}$ is a holomorphic and 1 - 1 function and $r > 1$, then for $\Omega = \phi(D(0, 1))$ and J any compact subset of $\partial\Omega$ it holds $S(\Omega, J) \neq \emptyset$ (and therefore, the class $S(\Omega, J)$ is G_δ - dense in $A(\Omega) = R(\Omega)$). Under the previous assumptions it holds $\phi'(e^{i\theta}) \neq 0$ for every $\theta \in \mathbb{R}$. We note that the same result holds if Ω is a Jordan domain and $\phi : D(0, 1) \rightarrow \Omega$ is holomorphic, 1 - 1 and onto, ϕ' extends continuously on $\overline{D(0, 1)}$ and $\phi'(e^{i\theta}) \neq 0$ for every $\theta \in \mathbb{R}$. We remind that ϕ extends to a homeomorphism from $\overline{D(0, 1)}$ to $\overline{\Omega}$, according to Osgood - Caratheodory Theorem [4]. The proof that $S(\Omega, J) \neq \emptyset$ is similar to the proof of Example 4.10 and therefore is omitted.

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